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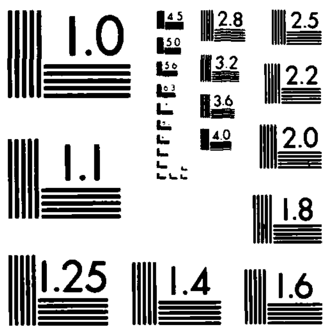

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ASYMPTOTIC BEHAVIORS OF THE SOLUTION OF AN ELLIPTIC EQUATION WITH PENALTY TERMS

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ABSTRACT

We study the boundary value problem for an elliptic equation with penalty terms. This problem approximates the boundary value problems with three types of homogeneous boundary conditions; i) the Dirichlet boundary condition, ii) the Neumann boundary condition, iii) the mixed boundary condition. We discuss asymptotic behaviors of the solutions of the above mentioned problems as the coefficient of the penalty term tends to zero. By using one of these properties, we can approximate the outward normal derivative defined on the boundary of the approximated problem prescribed with the Dirichlet condition, which is efficiently available to obtain the numerical solution of free boundary problems of various types.

AMS (MOS) Subject Classifications: 35J05, 35J67, 34E05, 34E99.

Key Words: Elliptic boundary value problems with discontinuous coefficients, Asymptotic expansions, Penalty methods.

Work Unit Number 1 - Applied Analysis

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1. Introduction

Let Ω_0 be connected domain in R^2 with smooth boundary Γ. Take Ω so as to satisfy (i) $\Omega \supset \bar{\Omega}_0$; (ii) $\Omega_1 = \Omega - \Omega_0$ is a connected domain; (iii) the measure of $\partial\Omega_1^{(1)} \cap \partial\Omega$ is positive or Ω_1 is unbounded; (iv) $\partial\Omega$ is smooth (see Fig.1).

We shall consider the boundary value problem defined in Ω for every $\varepsilon > 0$ and $\alpha, \beta \in R$.

Find $\psi^\varepsilon = \begin{cases} \psi_0^\varepsilon & \text{in } \Omega_0, \\ \psi_1^\varepsilon & \text{in } \Omega_1 \end{cases}$ such that

$$-\Delta \psi_0^\varepsilon + \lambda_0 \psi_0^\varepsilon = f \quad \text{in } \Omega_0 \tag{1.1}$$

$$-\varepsilon^{2\alpha} \cdot \Delta\psi_1^\varepsilon + \varepsilon^{-2\beta} \cdot \psi_1^\varepsilon = 0 \quad \text{in } \Omega_1 \tag{1.2}$$

$$\psi_0^\varepsilon = \psi_1^\varepsilon \quad \text{on } \Gamma \tag{1.3}$$

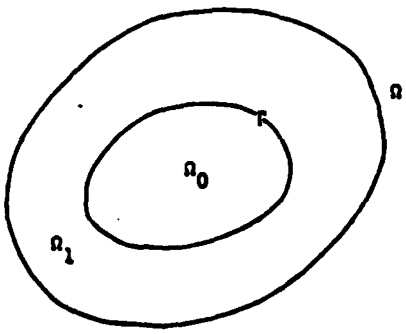

Fig.1

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(1) $\partial\Omega$ stands for the boundary of Ω.

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$$(1.4) \qquad \frac{\partial\psi_0^\varepsilon}{\partial n} = \varepsilon^{2\alpha}\cdot\frac{\partial\psi_1^\varepsilon}{\partial n} \quad \text{on } \Gamma$$

$$(1.5) \qquad \psi_1^\varepsilon = 0 \quad \text{on } \partial\Omega$$

$$\text{and} \quad \psi_1^\varepsilon \to 0 \quad (|x| = \sqrt{x_1^2 + x_2^2} \to \infty).$$

Here n is the outward normal on Γ to Ω_0 and λ_0 is a positive constant. It is found in Lions ([3], Chapter 1, p.80) that the boundary condition which the limit function of ψ_0^ε as $\varepsilon \to 0$ satisfies on Γ is classified into three types, which depend upon the relative value of α and β.

In this paper, we study an asymptotic behavior of ψ^ε on Γ when ε is small enough. We now summarize the contents of this paper. Section 2 includes four Theorems. In section 3, we prepare some Lemmas for the proofs of Theorems. Sections from 4 to 7 are devoted to the proofs of Theorems.

2. Theorems

2.1 We put

$$(2.1) \qquad K = \{\psi \in H^1(\Omega) \mid \psi = 0 \ \text{ on } \partial\Omega \ \text{ and } \ \psi \to 0 \ (|x| \to \infty)\}.$$

Then (1.1)-(1.5) are reformulated as follows:

Find $\psi^\varepsilon \in K$ such that

$$(2.2) \qquad \int_{\Omega_0} \nabla\psi^\varepsilon \nabla v dx + \lambda_0 \int_{\Omega_0} \psi^\varepsilon v dx + \varepsilon^{2\alpha}\cdot\int_{\Omega_1} \nabla\psi^\varepsilon \nabla v dx + \varepsilon^{-2\beta}\cdot\int_{\Omega_1} \psi^\varepsilon v dx$$

$$= \int_{\Omega_0} f v dx, \qquad \forall v \in K.$$

There exists a unique solution $\psi^\varepsilon (\in K)$ of (2.2) for $\forall f \in H^{-1}(\Omega_0)$. Putting $v = \psi^\varepsilon$ in (2.2), we see that ψ_0^ε is uniformly bounded in ε:

$$(2.3) \qquad \|\psi^\varepsilon\|^{(2)}_{1,\Omega_0} \le C < +\infty$$

(2) $\|v\|_{m,E}$ stands for the norm of v in $H^m(E)$.

where C depends upon only the data f.

When ε tends to zero, we can extract a sequence ε_n $(n = 1, 2, \ldots)$ such that

$$\psi_0^{\varepsilon_n} \to \psi_0^0 \qquad \text{weakly in } H^1(\Omega_0). \tag{2.4}$$

Then

$$\psi_0^{\varepsilon_n} \to \psi_0^0 \qquad \text{strongly in } H^s(\Omega_0) \qquad (\forall s < 1,\ [5]). \tag{2.5}$$

Let $v \in H_0^1(\Omega_0)$ and $\tilde{v}$ be the zero extension of v to Ω. Passing to the limit in (2.2) for $\tilde{v} \in H^1(\Omega)$ yields

$$\int_{\Omega_0} \nabla\psi_0^0 \nabla v\, dx + \lambda_0 \int_{\Omega_0} \psi_0^0 v\, dx = \int_{\Omega_0} f v\, dx, \qquad \forall v \in H_0^1(\Omega_0). \tag{2.6}$$

from which, we have

$$-\Delta\psi_0^0 + \lambda_0 \psi_0^0 = f \qquad \text{in } H^{-1}(\Omega_0). \tag{2.7}$$

If we assume $f \in H^{m-1}(\Omega_0)$ $(m \geq 0)$, then we have

$$\psi_0^0 \in H^{m+1}(\Omega_0). \tag{2.8}$$

By the trace theorem (Nečas [5]),

$$\psi_0^0\Big|_\Gamma \in H^{m+\frac{1}{2}}(\Gamma) \quad \text{and} \quad \frac{\partial\psi_0^0}{\partial n}\Big|_\Gamma \in H^{m-\frac{1}{2}}(\Gamma). \tag{2.9}$$

Moreover, ψ_0^0 satisfies on Γ:

<u>Theorem 1</u>(3) Suppose $f \in H^{m-1}(\Omega_0)$ $(m \geq 0)$.

(a) If $\beta > |\alpha|$, then

$$\psi_0^0\Big|_\Gamma = 0 \qquad \text{in } H^{m+\frac{1}{2}}(\Gamma). \tag{2.10}$$

---

(3) This theorem was proved in [3] for the case $\alpha > 0$ and $\beta > 0$. In this paper, we give another proof, which is simpler than in [3].

(b) If $\beta = \alpha > 0$, then

$$(2.11) \qquad \left(\frac{\partial \psi_0^0}{\partial n} + \psi_0^0\right)\Big|_\Gamma = 0 \qquad \text{in } H^{m-\frac{1}{2}}(\Gamma).$$

(c) If $\beta < \alpha$ and $\alpha > 0$, then

$$(2.12) \qquad \frac{\partial \psi_0^0}{\partial n}\Big|_\Gamma = 0 \qquad \text{in } H^{m-\frac{1}{2}}(\Gamma).$$

<u>Remark 1</u> There also holds $\psi_0^0\big|_\Gamma = 0$ in the case $\alpha + \beta \leq 0$ and $\alpha < 0$ under the same assumption.

2.2 We now state our main result as follows.

<u>Theorem 2</u> Suppose $f \in H^m(\Omega_0)$ $(m \geq 0)$ and let ε be small enough.

(a) If $\beta > |\alpha|$, then

$$(2.13) \qquad \psi^\varepsilon|_\Gamma = -\varepsilon^{\beta-\alpha} \cdot \frac{\partial \psi_0^0}{\partial n}\Big|_\Gamma + O(\varepsilon^{2(\beta-\alpha)} + \varepsilon^{\beta-3\alpha} + \varepsilon^{2(\alpha+\beta)}) \qquad \text{in } H^{m-\frac{1}{2}}(\Gamma)$$

where ψ_0^0 satisfies (2.10).

(b) If $\beta = \alpha > 0$, then

$$(2.14) \qquad \psi^\varepsilon|_\Gamma = \psi_0^0\big|_\Gamma + O(\varepsilon^{4\alpha}) \qquad \text{in } H^{m+\frac{1}{2}}(\Gamma)$$

where ψ_0^0 satisfies (2.11).

(c) If $|\beta| < \alpha$, then

$$(2.15) \qquad \psi^\varepsilon|_\Gamma = \psi_0^0\big|_\Gamma + O(\varepsilon^{\alpha-\beta}) \qquad \text{in } H^{m+\frac{1}{2}}(\Gamma)$$

where ψ_0^0 satisfies (2.12).

2.3 By using (a) of Theorem 2, we have the regularity results about ψ^ε.

__Theorem 3__ Suppose $f \in H^k(\Omega_0)$ $(k \geq 5)$ and let ε be small enough.

If $\beta > |\alpha|$, then

$$(2.16) \qquad \| \psi^\varepsilon \|_{W^{0,\infty}(\Gamma)} \leq O(\varepsilon^{\beta-\alpha}),$$

$$(2.17) \qquad \| \psi^\varepsilon \|_{W^{1,\infty}(\Omega)} \leq O(1+\varepsilon^{-2\alpha}),$$

$$(2.18) \qquad \| \psi^\varepsilon \|_{W^{2,\infty}(\Omega_1)} \leq O(\varepsilon^{-(3\alpha+\beta)}).$$

2.4 The motivation of this paper consists in __the integrated penalty method__ presented by one of the author [2]. The mathematical justification of this method was done in the sense of distribution. If we use (a) of Theorem 2, we can prove the key-point of this method in the framework of the Soborev space.

__Theorem 4__ Suppose $f \in H^m(\Omega_0)$ $(m \geq 0)$ and let ε be small enough.

If $\beta > |\alpha|$, then

$$(2.19) \qquad \| \varepsilon^{-2\beta} \cdot \int_{\Gamma^\perp(s)} \psi_1^\varepsilon d\Gamma^\perp + \frac{\partial \psi_0^0}{\partial n}(s) \|_{m-\frac{1}{2},\Gamma} = O(\varepsilon^{2(\alpha+\beta)}).$$

Here s stands for the length of the arc along Γ.

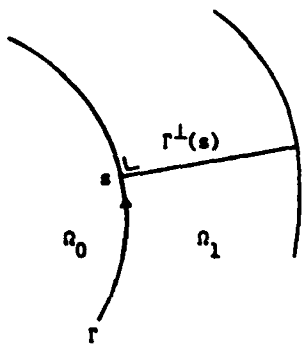

__Fig.2__

3. Preliminaries

The aim of this section is to give some preparatory lemmas which will be needed in the proofs of Theorems.

3.1 We first introduce some operators defined between traces on Γ.

(i) Define the mapping

$$(3.1) \qquad T_f : H^{\frac{1}{2}}(\Gamma) \ni a \to \left.\frac{\partial \psi_a}{\partial n}\right|_\Gamma \in H^{-\frac{1}{2}}(\Gamma):$$

ψ_a is the solution of the problem;

$$(3.2) \qquad -\Delta\psi + \lambda_0\psi = f \qquad \text{in } \Omega_0$$

$$(3.3) \qquad \psi|_\Gamma = a$$

where $f \in H^{m-1}(\Omega_0)$ $(m \geq 0)$.

(ii) Define the mapping

$$(3.4) \qquad R^\varepsilon : H^{-\frac{1}{2}}(\Gamma) \ni b \to \psi_b^\varepsilon|_\Gamma \in H^{\frac{1}{2}}(\Gamma);$$

ψ_b^ε is the solution of (3.2) with $f \equiv 0$ and the boundary condition

$$(3.5) \qquad \left.\left(\varepsilon^{\beta-\alpha} \cdot \frac{\partial\psi}{\partial n} + \psi\right)\right|_\Gamma = b.$$

(iii) Define the mapping

$$(3.6) \qquad S^\varepsilon : H^{\frac{1}{2}}(\Gamma) \ni a \to \left.\frac{\partial \psi_a^\varepsilon}{\partial n}\right|_\Gamma \in H^{-\frac{1}{2}} \quad ;$$

ψ_a^ε is the solution of the problem;

$$(3.7) \qquad -\varepsilon^{\alpha+\beta} \cdot \Delta\psi + \psi = 0 \qquad \text{in } \Omega_1$$

$$(3.8) \qquad \psi|_\Gamma = a$$

(3.9) $\psi|_{\partial\Omega} = 0$ and $\psi \to 0$ $(|x| \to \infty)$.

We denote T_f^m, S_m^ε and R_m^ε by the restriction of T_f, S^ε and R^ε to $H^{m+\frac{1}{2}}(\Gamma)$. But we abbreviate the suffix m hereafter.

3.2

__Lemma 1__ Let a, b be arbitrary in $H^{m+\frac{1}{2}}(\Gamma)$. Then

(3.10) $$T_f(a) - T_f(b) = T_0(a-b)$$

where T_0 implies $T_{f=0}$.

__Proof__ Let ψ_γ $(\gamma = a,b)$ be the solution of (3.2) under the boundary condition

(3.11) $$\psi|_\Gamma = \gamma.$$

Put $\Psi = \psi_a - \psi_b$. Ψ satisfies

(3.12) $$-\Delta\Psi + \lambda_0\Psi = 0 \quad \text{in } \Omega_0,$$

(3.13) $$\Psi|_\Gamma = a - b.$$

Then

(3.14) $$\left.\frac{\partial\Psi}{\partial n}\right|_\Gamma = T_0(a-b).$$

On the other hand,

(3.15) $$\left.\frac{\partial\Psi}{\partial n}\right|_\Gamma = \left.\frac{\partial\psi_a}{\partial n}\right|_\Gamma - \left.\frac{\partial\psi_b}{\partial n}\right|_\Gamma = T_f(a) - T_f(b).$$

From (3.14) and (3.15) follows (3.10). Here we should note that T_0 is linear and T_f is non-linear. ∎

__Lemma 2__

T_f and S are homeomorphic from $H^{m-\frac{1}{2}}(\Gamma)$ to $H^{m+\frac{1}{2}}(\Gamma)$ and R is homeomorphic from $H^{m+\frac{1}{2}}(\Gamma)$ to $H^{m-\frac{1}{2}}(\Gamma)$ for any $m \geq 0$.

Proof

1° T_f is injective from $H^{m+\frac{1}{2}}(\Gamma)$ into $H^{m-\frac{1}{2}}(\Gamma)$. In fact, let $a,b \in H^{m+\frac{1}{2}}(\Gamma)$ $(a \neq b)$. Suppose $T_f(a) = T_f(b)$. Then, by (3.10)

$0 = T_f(a) - T_f(b) = T_0(a-b) \neq 0$

because of the strong maximum principle under the assumption $\lambda_0 > 0$. This is a contradiction.

2° T_f is surjective from $H^{m+\frac{1}{2}}(\Gamma)$ onto $H^{m-\frac{1}{2}}(\Gamma)$. In fact, we choose any $b \in H^{m+\frac{1}{2}}(\Gamma)$. Then, the following problem:

$$-\Delta\psi + \lambda_0\psi = f \qquad \text{in } \Omega_0 \tag{3.16}$$

$$\left.\frac{\partial\psi}{\partial n}\right|_\Gamma = b \tag{3.17}$$

has a unique solution $\psi_b \in H^{m+1}(\Omega_0)$ if $\lambda_0 > 0$, which satisfies

$$\psi_b\big|_\Gamma \in H^{m+\frac{1}{2}}(\Gamma) \qquad \text{and} \qquad b = T_f(\psi_b\big|_\Gamma). \tag{3.18}$$

3° It is checked that T_f and $(T_f)^{-1}$ are continuous between $H^{m+\frac{1}{2}}(\Gamma)$ and $H^{m-\frac{1}{2}}(\Gamma)$ (see [1]).

4° Summing up 1°, 2° and 3°, we see that T_f is a homeomorphism from $H^{m+\frac{1}{2}}(\Gamma)$ onto $H^{m-\frac{1}{2}}(\Gamma)$. The repeated use of the above arguments gives that $(R^\varepsilon)^{-1}$ and S^ε are also homeomorphic between $H^{m+\frac{1}{2}}(\Gamma)$ and $H^{m-\frac{1}{2}}(\Gamma)$. ∎

3.3 Here we give the estimates of the norm of R^ε and S^ε, which are crucial for the proof of Theorems 1 and 2.

Lemma 3 Let ε be small enough and suppose $\beta \geq \alpha$ and $m \geq 0$. Then

$$(3.19) \qquad \| R^{\varepsilon}(a) \|_{m+\frac{1}{2},\Gamma} = O(\varepsilon^{\alpha-\beta}) \| a \|_{m-\frac{1}{2},\Gamma}, \qquad \text{for } \forall a \in H^{m-\frac{1}{2}}(\Gamma)$$

$$(3.20) \qquad \| R^{\varepsilon}(a) \|_{m-\frac{1}{2},\Gamma} = O(1) \| a \|_{m-\frac{1}{2},\Gamma}, \qquad \text{for } \forall a \in H^{m-\frac{1}{2}}(\Gamma)$$

and

$$(3.21) \qquad \| R^{\varepsilon}(a) - a \|_{m-\frac{1}{2},\Gamma} = O(\varepsilon^{\beta-\alpha}) \| a \|_{m+\frac{1}{2},\Gamma} \qquad \text{for } \forall a \in H^{m+\frac{1}{2}}(\Gamma).$$

__Proof__ Using Green's formula in the problem defining R^{ε}, we have

$$(3.22) \qquad \varepsilon^{\beta-\alpha} \cdot \int_{\Omega_0} (|\nabla\psi|^2 + \lambda_0 |\psi|^2) dx + \int_{\Gamma} |\psi|^2 ds = \int_{\Gamma} a\psi ds.$$

From (3.22) it follows

$$(3.23) \qquad \|\psi\|_{0,\Gamma} \leq \|a\|_{0,\Gamma}$$

and

$$(3.24) \qquad \varepsilon^{\beta-\alpha} \|\psi\|_{\frac{1}{2},\Gamma} \leq \|a\|_{-\frac{1}{2},\Gamma}.$$

Using the standard technique to raise up the regularity property of the solution of partial differential equations, we obtain (3.19) and (3.20).

Rewriting (3.5) with an aid of T_0 and R^{ε}, we have for $\forall a \in H^{m+\frac{1}{2}}(\Gamma)$

$$(3.25) \qquad \begin{aligned} \| R^{\varepsilon}(a) - a \|_{m-\frac{1}{2},\Gamma} &= \varepsilon^{\beta-\alpha} \| T_0 R^{\varepsilon}(a) \|_{m-\frac{1}{2},\Gamma} \\ &= O(\varepsilon^{\beta-\alpha}) \| R^{\varepsilon}(a) \|_{m+\frac{1}{2},\Gamma} \\ &= O(\varepsilon^{\beta-\alpha}) \| a \|_{m+\frac{1}{2},\Gamma} \qquad \text{(by 3.20)}. \end{aligned}$$

Here we have used the continuity of T_0 from $H^{m+\frac{1}{2}}(\Gamma)$ to $H^{m-\frac{1}{2}}(\Gamma)$. ∎

<u>Lemma 4</u> Let ε be small enough and $m \geq 0$.

(a) If $\sigma = \alpha + \beta + \rho(\alpha - \beta) > 0$ $(\rho \in R)$, then

$$(3.26) \qquad \| \varepsilon^{\sigma} S^{\varepsilon}(a) + \varepsilon^{\rho(\alpha-\beta)} a \|_{m-\frac{1}{2},\Gamma} = O(\varepsilon^{\sigma}) \| a \|_{m+\frac{1}{2},\Gamma}, \qquad \forall a \in H^{m+\frac{1}{2}}(\Gamma).$$

(b) If $\alpha + \beta > 0$, then

$$(3.27) \qquad \| S^{\varepsilon}(a) + \varepsilon^{-(\alpha+\beta)} a \|_{m-\frac{1}{2},\Gamma} = O(\varepsilon^{\alpha+\beta}) \| a \|_{m+\frac{1}{2},\Gamma}, \qquad \forall a \in H^{m+\frac{1}{2}}(\Gamma)$$

$$(3.28) \qquad \| (S^{\varepsilon})^{-1}(b) \|_{m+\frac{1}{2},\Gamma} = O(1) \| b \|_{m-\frac{1}{2},\Gamma}, \qquad \forall b \in H^{m-\frac{1}{2}}(\Gamma)$$

$$(3.29) \qquad \| (S^{\varepsilon})^{-1}(b) + \varepsilon^{\alpha+\beta} b \|_{m-\frac{1}{2},\Gamma} = O(\varepsilon^{3(\alpha+\beta)}) \| b \|_{m+\frac{1}{2},\Gamma}, \qquad \forall b \in H^{m+\frac{1}{2}}(\Gamma).$$

<u>Proof</u> We prove this lemma in the two cases. In the first case, we prove the special case $\Omega_1 = R^2_+ = \{(x_1,x_2) \mid 0 < x_1, -\infty < x_2 < +\infty\}$ $(\Omega_0 = R^2_-)$ by using the fourier transformation. Subsequently, we give the plan of the proof in the general geometry.

1° Let

$$\hat{\psi}^{\varepsilon}(x_1, \xi) = \int_{-\infty}^{\infty} e^{2\pi i \xi x_2} \cdot \psi^{\varepsilon}(x_1, x_2) dx_2$$

and

$$\hat{a}(\xi) = \int_{-\infty}^{\infty} e^{2\pi i \xi x_2} \cdot a(x_2) dx_2.$$

Here $\psi^\varepsilon(x_1,x_2)$ is the solution of the problem (3.7)-(3.9). Then $\hat{\psi}^\varepsilon$ satisfies

$$(3.30) \qquad -\frac{\partial^2\hat{\psi}^\varepsilon}{\partial x_1^2} + \varepsilon^{-2(\alpha+\beta)}\cdot(1+4\pi^2|\xi|^2\varepsilon^{2(\alpha+\beta)})\hat{\psi}^\varepsilon = 0 \qquad \text{in } R^2_+,$$

$$(3.31) \qquad \hat{\psi}^\varepsilon|_{x_1=0} = \hat{a}.$$

Solving (3.30) and (3.31), we have

$$(3.32) \qquad \hat{\psi}^\varepsilon = \hat{a}\cdot\exp\{-\varepsilon^{-(\alpha+\beta)}\cdot(1+4\pi^2\cdot|\xi|^2\varepsilon^{2(\alpha+\beta)})^{\frac{1}{2}}\, x_1\}.$$

From (3.32)

$$(3.33) \qquad \left.\frac{\partial\hat{\psi}^\varepsilon}{\partial x_1}\right|_{x_1=0} = \widehat{S^\varepsilon(a)} = -\varepsilon^{-(\alpha+\beta)}\cdot(1+4\pi^2\cdot|\xi|^2\cdot\varepsilon^{2(\alpha+\beta)})^{\frac{1}{2}}\cdot\hat{a}.$$

We compute

$$(3.34) \qquad \varepsilon^\sigma\cdot\widehat{S^\varepsilon(a)} + \varepsilon^{\rho(\alpha-\beta)}\cdot\hat{a}$$

$$= \varepsilon^{\rho(\alpha+\beta)}\{1-(1+4\pi^2\cdot|\xi|^2\cdot\varepsilon^{2(\alpha+\beta)})^{\frac{1}{2}}\}\hat{a}$$

$$= -4\pi^2\cdot\varepsilon^\sigma\cdot\frac{|\xi|^2\cdot\varepsilon^{\alpha+\beta}\hat{a}}{1+(1+4\pi^2\cdot|\xi|^2\cdot\varepsilon^{2(\alpha+\beta)})^{\frac{1}{2}}}.$$

From (3.34), it follows

$$(3.35) \qquad \int_{-\infty}^{\infty}(1+4\pi^2\cdot|\xi|^2)^{m-\frac{1}{2}}\left|\varepsilon^\sigma\cdot\widehat{S^\varepsilon(a)} + \varepsilon^{\rho(\alpha-\beta)}\cdot\hat{a}\right|^2 d\xi$$

$$= 16\pi^4\cdot\varepsilon^{2\sigma}\int_{-\infty}^{\infty}\frac{(1+4\pi^2|\xi|^2)^{m-\frac{1}{2}}\cdot|\xi|^2\cdot\varepsilon^{2(\alpha+\beta)}\cdot|\xi|^2\cdot|\hat{a}|^2}{\{1+(1+4\pi^2\cdot|\xi|^2\varepsilon^{2(\alpha+\beta)})^{\frac{1}{2}}\}^2}\, d\xi$$

$$\leq O(\varepsilon^{2\sigma})\int_{-\infty}^{\infty}(1+4\pi^2\cdot|\xi|^2)^{m+\frac{1}{2}}\cdot|\hat{a}|^2\,d\xi.$$

Hence we obtain

$$(3.36)\qquad \|\varepsilon^{\sigma}\cdot S^{\varepsilon}(a)+\varepsilon^{\rho(\alpha-\beta)}\cdot a\|_{m-\frac{1}{2},\Gamma}\leq O(\varepsilon^{\sigma})\|a\|_{m+\frac{1}{2},\Gamma}\,.$$

Repeating the simular arguments as above, we conclude (3.27)-(3.29).

2° Let us now deal with the <u>general case</u>. The domain Ω_1 is a regular simply connected domain; then there exists a (fixed) regular conformal mapping $w=f(z)=u_1+iu_2$ $(z=x_1+ix_2)$ which maps Ω_1 into R_+^2. As a matter of fact, Γ is mapped into the u_2-axis of w-plane. Then the transformed solution $\Psi^{\varepsilon}=\psi^{\varepsilon}(f^{-1}(w))$ satisfies

$$(3.37)\qquad -\varepsilon^{2(\alpha+\beta)}\cdot\Delta\Psi^{\varepsilon}+\left|\frac{dz}{dw}\right|^2\Psi^{\varepsilon}=0\qquad \text{in}\quad R_+^2,$$

$$(3.38)\qquad \Psi^{\varepsilon}|_{u_1=0}=A(u_2)=a(f^{-1}(w)).$$

By means of the iterative method proposed in the theory of singular perturbation (see [3]), Ψ^{ε} is asymptotically developed in the following way:

$$(3.39)\qquad \psi^{\varepsilon}=\psi_{\varepsilon}^0+\varepsilon^{\alpha+\beta}\psi_{\varepsilon}^1+\varepsilon^{2(\alpha+\beta)}\psi_{\varepsilon}^2+\dots+\varepsilon_{\varepsilon}^{n(\alpha+\beta)}\psi^n+w_{\varepsilon}.$$

Using (3.39), we obtain (3.26)-(3.29) (see the appendix). ▮

3.4 Define

$$(3.40)\qquad \varphi^{\varepsilon}=\psi^{\varepsilon}|_{\Gamma}\in H^{m+\frac{1}{2}}(\Gamma),$$

$$(3.41)\qquad \varphi^{0}=\psi_0^0\Big|_{\Gamma}\in H^{m+\frac{1}{2}}(\Gamma).$$

Then we have

<u>Lemma 5</u> Let $\varepsilon_n\to 0$ $(n=1,2,\dots)$. Then

(3.42) $$\varphi^{\epsilon_n} \to \varphi^0 \qquad \text{weakly in } H^{\frac{1}{2}}(\Gamma),$$

(3.43) $$T_\epsilon(\varphi^{\epsilon_n}) \to T_\epsilon(\varphi^0) \qquad \text{weakly in } H^{-\frac{1}{2}}(\Gamma).$$

Proof Recalling (2.4), (3.42) is obvious. Let a, b be arbitrary in $H^{\frac{1}{2}}(\Gamma)$. Then, we denote by ψ_γ $(\gamma = a, b)$ the solution of the problem:

(3.44) $$-\Delta\psi + \lambda_0\psi = 0 \qquad \text{in } \Omega_0,$$

(3.45) $$\psi|_\Gamma = \gamma.$$

By using Green's formula, we have

(3.46) $$\int_\Gamma b\,T_0(a)ds - \int_\Gamma a\,T_0(b)ds = \int_{\Omega_0}(\psi_b\Delta\psi_a - \psi_a\Delta\psi_b)dx = 0.$$

Using (3.10) and taking $a = \varphi^{\epsilon_n} - \varphi^0$ in (3.46),

(3.47) $$\int_\Gamma \{T_\epsilon(\varphi^{\epsilon_n}) - T_\epsilon(\varphi^0)\}\,b\,ds = \int_\Gamma T_0(\varphi^{\epsilon_n} - \varphi^0)\,b\,ds$$

$$= \int_\Gamma (\varphi^{\epsilon_n} - \varphi^0)T_0(b)ds \to 0 \qquad (\epsilon_n \to 0).$$

Lemma 6 Suppose $\sigma = \alpha + \beta + \rho(\alpha - \beta) > 0$ $(\rho \in R)$. Then

(3.48) $$\epsilon_n^{(\rho-1)(\alpha-\beta)}\cdot T_\epsilon(\varphi^{\epsilon_n}) + \epsilon_n^{\rho(\alpha-\beta)}\cdot\varphi^{\epsilon_n} \to 0$$

$$\text{strongly in } H^{-\frac{1}{2}}(\Gamma) \quad \text{as } \epsilon_n \to 0.$$

Proof By using the definition of T_ϵ and S^ϵ, (1.4) is rewritten as follows:

(3.49) $$T_\epsilon(\varphi^\epsilon) = \epsilon^{2\alpha}\cdot S^\epsilon(\varphi^\epsilon).$$

Taking $a = \varphi^{\epsilon_n}$ and $\epsilon = \epsilon_n$ in (3.26) for $m = 0$ and substituting (3.49), we have

(3.50) $$\| \epsilon_n^{(\rho-1)(\alpha-\beta)}\cdot T_\epsilon(\varphi^{\epsilon_n}) + \epsilon_n^{\rho(\alpha-\beta)}\cdot\varphi^{\epsilon_n} \|_{-\frac{1}{2},\Gamma} = O(\epsilon_n^\sigma)\|\varphi^{\epsilon_n}\|_{\frac{1}{2},\Gamma}.$$

Let $\varepsilon_n \to 0$. Then we conclude (3.48) with an aid of (2.3). ∎

4. Proof of Theorem 1

(a) Let $\rho = 0$ and $\beta > |\alpha|$ in the assumption of Lemma 6. Then $\sigma = \alpha + \beta > 0$ and (3.48) becomes

$$(4.1) \qquad \varepsilon_n^{\beta-\alpha} \cdot T_f(\varphi^{\varepsilon_n}) + \varphi^{\varepsilon_n} \to 0 \qquad \text{strongly in } H^{-\frac{1}{2}}(\Gamma).$$

By (3.42) and (3.43),

$$(4.2) \qquad \varphi^0 \quad (= \psi_0^0\big|_\Gamma) = 0.$$

(b) Let $\alpha = \beta > 0$. Then $\sigma = 2\alpha > 0$ and (3.48) becomes

$$(4.3) \qquad T_f(\varphi^{\varepsilon_n}) + \varphi^{\varepsilon_n} \to 0 \qquad \text{strongly in } H^{-\frac{1}{2}}(\Gamma)$$

which implies

$$(4.4) \qquad T_f(\varphi^0) + \varphi^0 = \left(\frac{\partial \psi_0^0}{\partial n} + \psi_0^0\right)\Big|_\Gamma = 0$$

by the definition of T_f and φ^0.

(c) Let $\rho = 1$, $\alpha > \beta$ and $\alpha > 0$. Then $\sigma = 2\alpha > 0$ and (3.48) becomes

$$(4.5) \qquad T_f(\varphi^{\varepsilon_n}) + \varepsilon_n^{\alpha-\beta} \cdot \varphi^{\varepsilon_n} \to 0 \qquad \text{strongly in } H^{-\frac{1}{2}}(\Gamma)$$

from which

$$(4.6) \qquad T_f(\varphi^0) \ \left(= \frac{\partial \psi_0^0}{\partial n}\Big|_\Gamma\right) = 0 \qquad \text{in } H^{-\frac{1}{2}}(\Gamma).$$

Combining (2.8) with the results obtained above, we conclude (2.10)-(2.12). ∎

5. Proof of Theorem 2

5.1 Using (3.49), the problem (1.1)-(1.5) is transformed into the following one:

Find $a \in H^{m+\frac{1}{2}}(\Gamma)$ such that

$$T_f(a) = \epsilon^{2\alpha} \cdot S^{\epsilon}(a). \tag{5.1}$$

Hereafter we call (5.1) the <u>transmission equation</u>. As a matter of fact, the solution of (5.1) is equal to the trace $\phi^{\epsilon}|_{\Gamma}$ of the solution of the problem (1.1)-(1.5).

5.2 Let b be arbitrary in $H^{m+\frac{1}{2}}(\Gamma)$. Then, combining (5.1) and (3.10), we have

$$T_0(a-b) - \epsilon^{2\alpha} \cdot S^{\epsilon}(a) = -T_f(b) \in H^{m-\frac{1}{2}}(\Gamma). \tag{5.2}$$

Let us begin to prove (a), in which $\beta > |\alpha|$ is assumed. By (a) of Theorem 1, we have $\phi_0^0\big|_{\Gamma} = 0$. Therefore we choose $b = 0$ in (5.2). On substituting (3.27) into (5.2), we get

$$T_0(a) + \epsilon^{\alpha-\beta} \cdot a - \epsilon^{2\alpha} \cdot S_1(a) = -T_f(0), \qquad \text{or} \tag{5.3}$$

$$\epsilon^{\beta-\alpha} \cdot T_0(a) + a = \epsilon^{\alpha+\beta} \cdot S_1^{\epsilon}(a) - \epsilon^{\beta-\alpha} \cdot T_f(0)$$

where $\quad S_1^{\epsilon}(a) = S^{\epsilon}(a) + \epsilon^{-(\alpha+\beta)} \cdot a.$

The definition of R^{ϵ} allows us to rewrite (5.3) by

$$a = \epsilon^{\alpha+\beta} \cdot R^{\epsilon} S_1^{\epsilon}(a) - \epsilon^{\beta-\alpha} \cdot R^{\epsilon}(T_f(0)). \tag{5.4}$$

Let $\epsilon(>0)$ be small enough in (5.4).

Using Lemmas 2, 3 and 4, we see that the mapping $\epsilon^{\alpha+\beta} \cdot R^{\epsilon} S_1^{\epsilon}$ becomes the contraction mapping from $H^{m+\frac{1}{2}}(\Gamma)$ onto itself if ϵ is small enough and $\beta > \alpha > -\frac{1}{3}$.

Indeed,

$$(5.5)\qquad \epsilon^{\alpha+\beta}\| R^{\epsilon}S_1^{\epsilon}(a)\|_{m+\frac{1}{2},\Gamma} \leq O(\epsilon^{2\alpha})\| S_1^{\epsilon}(a)\|_{m-\frac{1}{2},\Gamma} \qquad \text{(by 3.19)}$$

$$\leq O(\epsilon^{3\alpha+\beta})\, \| a \|_{m+\frac{1}{2},\Gamma} . \qquad \text{(by 3.27)}$$

On the other hand, by (3.21)

$$(5.6)\qquad R^{\epsilon}(T_f(0)) = T_f(0) + O(\epsilon^{\beta-\alpha}) \qquad \text{in } H^{m-\frac{1}{2}}(\Gamma).$$

Here we note that $T_f(0)$ should be included in $H^{m+\frac{1}{2}}(\Gamma)$. Therefore, we have to assume $f \in H^{m}(\Omega_0)$.

Summing up (5.4), (5.5) and (5.6), we have

$$(5.7)\qquad a = \psi^{\epsilon}|_{\Gamma} = (I - \epsilon^{\alpha+\beta}\cdot R^{\epsilon}S_1^{\epsilon})^{-1}\{-\epsilon^{\beta-\alpha}\cdot T_f(0) + O(\epsilon^{2(\beta-\alpha)})\}$$

$$= -\epsilon^{\beta-\alpha}\cdot T_f(0) + O(\epsilon^{2(\beta-\alpha)} + \epsilon^{2(\alpha+\beta)}) \qquad \text{in } H^{m-\frac{1}{2}}(\Gamma)$$

if $\beta > \alpha > -\frac{1}{3}\beta$.

We remove into the case $-\alpha < \beta \leq -3$. Operating $\epsilon^{-2\alpha}\cdot(S^{\epsilon})^{-1}$ on both sides of (5.2), we have

$$(5.8)\qquad a = \epsilon^{-2\alpha}(S^{\epsilon})^{-1}T_0(a) + \epsilon^{-2\alpha}\cdot(S^{\epsilon})^{-1}(T_f(0)).$$

Let ϵ (> 0) be small enough. Then $\epsilon^{-2\alpha}\cdot(S^{\epsilon})^{-1}T_0$ becomes the contraction mapping from $H^{m+\frac{1}{2}}(\Gamma)$ onto itself if $\alpha < 0$ and $\alpha + \beta > 0$. In fact, the boundedness of T_0 and (3.28) yields

$$(5.9)\qquad \epsilon^{-2\alpha}\, \|(S^{\epsilon})^{-1}T_0(a)\|_{m+\frac{1}{2},\Gamma} \leq O(\epsilon^{-2\alpha})\, \|a\|_{m+\frac{1}{2},\Gamma} .$$

Therefore, by using (5.8), (5.9) and (3.29), we have

$$a = \psi^\varepsilon|_\Gamma = \{I - \varepsilon^{-2\alpha}(S^\varepsilon)^{-1}T_0\}^{-1}\{-\varepsilon^{\beta-\alpha}T_f(0) + O(\varepsilon^{\alpha+3\beta})\} \tag{5.10}$$

$$= -\varepsilon^{\beta-\alpha}T_f(0) + O(\varepsilon^{\beta-3\alpha} + \varepsilon^{\alpha+3\beta}) \qquad \text{in } H^{m-\frac{1}{2}}(\Gamma),$$

if $\alpha + \beta > 0$ and $\alpha > 0$.

Combining (5.7) and (5.10), we obtain (2.13). ∎

5.3 We shall prove (b) of Theorem 2, in which $\alpha = \beta > 0$ is assumed. From (b) of Theorem 1 follows $T_f(\psi_0^0) + \psi_0^0 = 0$ on Γ. Choose $b = \psi_0^0|_\Gamma$ in (5.2). Then we have

$$T_0(a - \psi_0^0) - \varepsilon^{2\alpha}S^\varepsilon(a) = -T_f(\psi_0^0) = \psi_0^0\,. \tag{5.11}$$

By (3.27), we have

$$T_0(a - \psi_0^0) + a - \psi_0^0 = \varepsilon^{2\alpha}\cdot S_1^\varepsilon(a) \tag{5.12}$$

where $S_1^\varepsilon(a) = S^\varepsilon(a) + \varepsilon^{-2\alpha}a$. By use of R^ε with $\alpha = \beta$,

$$a - \psi_0^0 = \varepsilon^{2\alpha}R^\varepsilon S_1^\varepsilon(a). \tag{5.13}$$

Then $\varepsilon^{2\alpha}\cdot R^\varepsilon S_1^\varepsilon$ becomes the contraction mapping from $H^{m+\frac{1}{2}}(\Gamma)$ onto itself if $\alpha > 0$ and ε is small enough. In fact, by (3.19) and (3.27), we have

$$\varepsilon^{2\alpha}\,\|R^\varepsilon S_1^\varepsilon(a)\|_{m+\frac{1}{2},\Gamma} = O(\varepsilon^{4\alpha})\,\|a\|_{m+\frac{1}{2},\Gamma}\,. \tag{5.14}$$

Therefore we have

$$a = \psi^\varepsilon|_\Gamma = (I - \varepsilon^{2\alpha}R^\varepsilon S_1^\varepsilon)^{-1}\psi_0^0 = \psi_0^0 + O(\varepsilon^{4\alpha}) \qquad \text{in } H^{m+\frac{1}{2}}(\Gamma). \tag{5.15}$$ ∎

5.4 Now we are in the final step to prove (c). In this case, $\alpha > \beta$ and $\alpha > 0$ are assumed. (c) of Theorem 1 gives us $T_f(\psi_0^0) = 0$. Put $b = \psi_0^0|_\Gamma$ in (5.2). Then we have

$$T_0(a - \psi_0^0) - \varepsilon^{2\alpha}\cdot S^\varepsilon(a) = -T_f(\psi_0^0) = 0. \tag{5.16}$$

Operating $(T_0)^{-1}$ on both sides of (5.16), we have

$$(5.17) \qquad a - \psi_0^0 = \varepsilon^{2\alpha}(T_0)^{-1}S^\varepsilon(a - \psi_0^0) + \varepsilon^{2\alpha}(T_0)^{-1}S^\varepsilon(\psi_0^0).$$

Repeating the similar arguments as in the proofs of (a) and (b), $\varepsilon^{2\alpha}\cdot(T_0)^{-1}S^\varepsilon$ becomes the contraction mapping from $H^{m+\frac{1}{2}}(\Gamma)$ onto itself if $\alpha > \beta$ and ε is small enough. Then we have

$$(5.18) \qquad a = \psi^\varepsilon|_\Gamma = \psi_0^0 + (I - \varepsilon^{2\alpha}(T_0)S^\varepsilon)^{-1}(\varepsilon^{2\alpha}(T_0)^{-1}S^\varepsilon(\psi_0^0))$$

$$= \psi_0^0 + O(\varepsilon^{\alpha-\beta}) \qquad \text{in } H^{m+\frac{1}{2}}(\Gamma). \quad \blacksquare$$

6. Proof of Theorem 3

6.1 Assume $f \in H^k(\Omega_0)$ $(k \geq \frac{5}{2})$ and $\beta > |\alpha|$. Then, from (2.7) and (2.10), we have

$$(6.1) \qquad \psi_0^0 \in H_0^1(\Omega_0) \cap H^{k+2}(\Omega_0),$$

$$(6.2) \qquad \left.\frac{\partial\psi_0^0}{\partial n}\right|_\Gamma \in H^{k+\frac{1}{2}}(\Gamma) \cap C^{k-1,\delta}(\Gamma) \quad (0 < \delta < 1).$$

By using (2.13) and (6.2), we have

$$(6.3) \qquad \psi^\varepsilon|_\Gamma = O(\varepsilon^{\beta-\alpha}) \qquad \text{in } H^{k+\frac{1}{2}}(\Gamma) \cap C^{k-1,\delta}(\Gamma).$$

By applying the maximum principle to the problem (1.2), (1.3) and (1.5) and using (6.3), we obtain

$$(6.4) \qquad \| \psi_1^\varepsilon \|_{C(\Omega_1)} \leq O(\varepsilon^{\beta-\alpha}).$$

We compute on Γ;

$$(6.5) \qquad \| \psi_1^\varepsilon \|_{k-\frac{1}{2},\Gamma} \leq \| \frac{\partial\psi_1^\varepsilon}{\partial n} \|_{k-\frac{1}{2},\Gamma} + \| \frac{\partial\psi_1^\varepsilon}{\partial s} \|_{k-\frac{1}{2},\Gamma}$$

where s is the arc length of Γ.

By (6.3), we have

$$\left\| \frac{\partial \psi_1^\varepsilon}{\partial s} \right\|_{k-\frac{1}{2},\Gamma} \leq O(\varepsilon^{\beta-\alpha}). \tag{6.6}$$

From the definition of S^ε, we have

$$\left.\frac{\partial \psi_1^\varepsilon}{\partial n}\right|_\Gamma = S^\varepsilon(\psi^\varepsilon|_\Gamma) \in H^{k-\frac{1}{2}}(\Gamma). \tag{6.7}$$

By (3.27) and (2.13),

$$\left\| \frac{\partial \psi_1^\varepsilon}{\partial n} \right\|_{k-\frac{1}{2},\Gamma} \leq \varepsilon^{-(\alpha+\beta)} \cdot \|\psi_1^\varepsilon\|_{k+\frac{1}{2},\Gamma} \leq O(\varepsilon^{-2\alpha}). \tag{6.8}$$

Combining (6.6) and (6.8),

$$\|\nabla \psi_1^\varepsilon\|_{k-\frac{1}{2},\Gamma} \leq O(\varepsilon^{-2\alpha}). \tag{6.9}$$

Similarly, we have

$$\|\nabla \psi_1^\varepsilon\|_{k-\frac{1}{2},\partial\Omega} \leq \left\| \frac{\partial \psi^\varepsilon}{\partial n} \right\|_{k-\frac{1}{2},\partial\Omega} \leq O(1) \tag{6.10}$$

because of (6.4), $\psi_1^\varepsilon|_{\partial\Omega} = 0$ and $\left.\frac{\partial \psi_1^\varepsilon}{\partial n}\right|_{\partial\Omega} \in H^{k-\frac{1}{2}}(\partial\Omega) \cap C^{k-2,\delta}(\partial\Omega)$.
Put $\Psi^\varepsilon = \nabla \psi_1^\varepsilon$. Then Ψ^ε satisfies

$$-\varepsilon^{2(\alpha+\beta)} \Delta \Psi^\varepsilon + \Psi^\varepsilon = 0 \qquad \text{in } \Omega_1. \tag{6.11}$$

From the maximum principle together with (6.9) and (6.10), it follows

$$\|\nabla \psi^\varepsilon\|_{C(\bar{\Omega}_1)} \leq O(1 + \varepsilon^{-2\alpha}). \tag{6.12}$$

Here we have to assume $k \geq 4$ to obtain the good regularity of ψ^ε. Repeating the *similar argument, we have*

$$(6.13) \qquad \left\| \frac{\partial^2 \psi^\varepsilon}{\partial x_i \partial x_j} \right\|_{C(\overline{\Omega}_1)} \leq O(\varepsilon^{-(3\alpha+\beta)}) \qquad (k \geq 5).$$

7. Proof of Theorem 4

In the final section we give the proof of Theorem 4 under the drastic assumption. Suppose $\Omega_1 = R^2_+$.

In the same way as in 1° of the proof of Lemma 4, we transform ψ^ε_1 into $\widehat{\psi}^\varepsilon$. Then $\widehat{\psi}^\varepsilon$ satisfies

$$(7.1) \qquad \widehat{\psi}^\varepsilon(x_1, \xi) = \widehat{\psi}^\varepsilon|_{x_1=0} \cdot \exp(-\varepsilon^{-(\alpha+\beta)}(1 + 4\pi^2 |\xi|^2 \cdot \varepsilon^{2(\alpha+\beta)})^{\frac{1}{2}} x_1) \quad \text{in} \quad R^2_+.$$

By (2.13) of Theorem 2, we have

$$(7.2) \qquad \widehat{\psi}^\varepsilon\Big|_{x_1=0} = -\varepsilon^{\beta-\alpha} \cdot \widehat{\frac{\partial \psi^0_0}{\partial x_1}}\Big|_{x_1=0} + \cdots \qquad \text{in} \quad H^{m-\frac{1}{2}}(\Gamma).$$

By substituting (7.2) into (7.1), we have

$$(7.3) \qquad I(\xi) = \frac{1}{\varepsilon^{2\beta}} \cdot \int_0^\infty \widehat{\psi}^\varepsilon(x_1, \xi) dx_1$$

$$= -\widehat{\frac{\partial \psi^0_0}{\partial x_1}}\Big|_{x_1=0} \cdot \frac{1}{(1+4\pi^2 \cdot |\xi|^2 \cdot \varepsilon^{2(\alpha+\beta)})^{\frac{1}{2}}} + \cdots .$$

We compute

$$(7.4) \qquad I(\xi) + \widehat{\frac{\partial \psi^0_0}{\partial x_1}}\Big|_{x_1=0}$$

$$= 4\pi^2 \cdot \varepsilon^{2(\alpha+\beta)} \cdot \frac{|\xi|^2}{(1+4\pi^2 \cdot |\xi|^2 \cdot \varepsilon^{2(\alpha+\beta)})} \cdot \frac{1}{(1+(1+4\pi^2 \cdot |\xi|^2 \cdot \varepsilon^{2(\alpha+\beta)}))}$$

From (7.4), we have

$$(7.5) \qquad \left\| \frac{1}{\varepsilon^{2\beta}} \int_0^\infty \psi_1^\varepsilon \, dx_1 + \frac{\partial \psi_0^0}{\partial x_1} \right\|_{m-\frac{1}{2}, \Gamma} = O(\varepsilon^{2(\alpha+\beta)}).$$

Appendix

For simplicity, we assume $A(u_2) \in C_0^\infty(\hat{\Gamma})$ and rewrite $|\frac{dz}{dw}| = a(u_1, u_2)$.

1° Here we state how to construct ψ_ε^n $(n = 0, 1, 2, \ldots)$ in (3.39). Let ψ_ε^0 be the solution of the following ordinary differential equation:

$$(A.1) \qquad -\varepsilon^{2(\alpha+\beta)} \cdot \frac{\partial^2 \psi_\varepsilon^0}{\partial u_1^2} + a(0, u_2)^2 \psi_\varepsilon^0 = 0 \qquad \text{in } R_+^2,$$

$$(A.2) \qquad \psi_\varepsilon^0 \Big|_{u_1=0} = A(u_2) \qquad \text{and} \qquad \psi_\varepsilon^0 \to 0 \quad (u_2 \to +\infty).$$

Solving (A.1,2), we get

$$(A.3) \qquad \psi_\varepsilon^0 = A(u_2) \cdot \exp(-\varepsilon^{-(\alpha+\beta)} \cdot a(0, u_2) u_1)$$

We compute

$$(A.4) \qquad -\varepsilon^{2(\alpha+\beta)} \cdot \Delta \psi_\varepsilon^0 + a(u_1, u_2)^2 \psi_\varepsilon^0$$

$$= -\varepsilon^{2(\alpha+\beta)} \cdot \frac{d^2 A}{du_2^2} + \varepsilon^{-2(\alpha+\beta)} \cdot (a(0, u_2)^2$$

$$- a(u_1, u_2)^2) \cdot \exp(-\varepsilon^{-(\alpha+\beta)} \cdot a(0, u_2) u_1)$$

$$\equiv \varepsilon^{2(\alpha+\beta)} \cdot g_\varepsilon^0(u_1, u_2).$$

Let ψ_ε^1 be the solution of the problem:

(A.5) $$-\varepsilon^{2(\alpha+\beta)}\cdot\frac{\partial^2\psi_\varepsilon^1}{\partial u_1^2} + a(0,u_2)^2\psi_\varepsilon^1 = -g_\varepsilon^0(u_1,u_2) \qquad \text{in } \mathbb{R}_+^2$$

(A.6) $$\psi_\varepsilon^1\Big|_{u_1=0} = 0 \quad \text{and} \quad \psi_\varepsilon^1 \to 0 \quad (u_2 \to +\infty).$$

Solving (A.5,6) and computing

(A.7) $$-\varepsilon^{2(\alpha+\beta)}\cdot\Delta\psi_\varepsilon^1 + a(u_1,u_2)^2\cdot\psi_\varepsilon^1 = \varepsilon^{2(\alpha+\beta)}\cdot g_\varepsilon^1(u_1,u_2),$$

we can construct the equation which ψ_ε^2 satisfies in the following way:

(A.8) $$-\varepsilon^{2(\alpha+\beta)}\frac{\partial^2\psi_\varepsilon^2}{\partial u_1^2} + a(0,u_2)^2\psi_\varepsilon^2 = g_\varepsilon^1(u_1,u_2),$$

(A.9) $$\psi_\varepsilon^2\Big|_{u_1=0} = 0 \quad \text{and} \quad \psi_\varepsilon^2 \to 0 \quad (u_2 \to +\infty).$$

Using the cascade system defined above, we can obtain ψ_ε^n $(n = 0, 1, 2, \ldots)$.

2° We put

(A.10) $$\theta^\varepsilon = \psi_\varepsilon^0 + \varepsilon^{2(\alpha+\beta)}\cdot\psi_\varepsilon^1 + \cdots + \varepsilon^{2n(\alpha+\beta)}\cdot\psi_\varepsilon^n$$

and

(A.11) $$w_\varepsilon = \psi^\varepsilon - \theta^\varepsilon.$$

Then w_ε satisfies

(A.12) $$-\varepsilon^{2(\alpha+\beta)}\cdot\Delta w_\varepsilon + a(u_1,u_2)^2 w_\varepsilon = O(\varepsilon^{(n+2)(\alpha+\beta)}) \qquad \text{in } \mathbb{R}_+^2,$$

(A.13) $$w_\varepsilon\Big|_{u_1=0} = 0 \quad \text{and} \quad w_\varepsilon \to 0 \quad (u_2 \to +\infty).$$

From (A.12,13), we have

(A.14) $$\| w_\varepsilon \|_{L^2(R^2_+)} \leq O(\varepsilon^{(n+2)(\alpha+\beta)}),$$

(A.15) $$\| w_\varepsilon \|_{H^1(R^2_+)} \leq O(\varepsilon^{(n+1)(\alpha+\beta)})$$

and moreover

(A.16) $$\| w_\varepsilon \|_{H^{n+2}(R^2_+)} \leq O(1).$$

3° We compute

(A.17) $$\left.\frac{\partial\psi^\varepsilon}{\partial u_1}\right|_{u_1=0} = \left.\frac{\partial\theta^\varepsilon}{\partial u_1}\right|_{u_1=0} - \left.\frac{\partial w^\varepsilon}{\partial u_1}\right|_{u_1=0}$$

where

(A.18) $$\left.\frac{\partial\theta^\varepsilon}{\partial u_1}\right|_{u_1=0} = \sum_{k=0}^{n} \varepsilon^{2k(\alpha+\beta)} \cdot \left.\frac{\partial\psi^k_\varepsilon}{\partial u_1}\right|_{u_1=0} .$$

On the other hand, from (A.16)

(A.19) $$\left\| \left.\frac{\partial w_\varepsilon}{\partial u_1}\right|_{\hat\Gamma} \right\|_{H^{n+\frac{1}{2}}(\hat\Gamma)} \leq O(1) \qquad (\hat\Gamma\text{: the } u_2 \text{ axis of } w \text{ plane}).$$

If we choose $n+\frac{1}{2} \geq m-\frac{1}{2}$ (or $n \geq m-1$), then we have

(A.20) $$\left.\frac{\partial\psi^\varepsilon}{\partial u_1}\right|_{\hat\Gamma} = a(0\,,\,u_2)(-\varepsilon^{-(\alpha+\beta)} + S_1^\varepsilon)A(u_2) \in H^{m-\frac{1}{2}}(\hat\Gamma)$$

for any $A \in C_0^\infty(\hat\Gamma)$.

By noting $\left.\frac{\partial\psi^\varepsilon}{\partial n}\right|_\Gamma = \frac{1}{a(0,u_2)} \cdot \left.\frac{\partial\psi^\varepsilon}{\partial u_1}\right|_{\hat\Gamma}$ and using the density argument, we conclude (3.26)-(3.29).

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We study the boundary value problem for an elliptic equation with penalty terms. This problem approximates the boundary value problems with three types of homogeneous boundary conditions; i) the Dirichlet boundary condition, ii) the Neumann boundary condition, iii) the mixed boundary condition. We discuss asymptotic behaviors of the solutions of the above mentioned problems as the coefficient of the penalty term tends to zero. By using one of these properties, we can approximate the outward normal derivative defined on the boundary of the approximated problem prescribed with the Dirichlet condition, which is efficiently available to obtain the numerical solution of free boundary problems of various types.

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